Pergamon

Int. J. Heat Mass Transfer. Vol. 37, No. 17, pp. 2737–2743, 1994

0017-9310/94 $7.00+0.00

0017-9310(94)E0085-D

The thermal stresses and displacements in a two-dimensional convective half-space for a moving heat source

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(*Received* 23 *June* 1993 *and in final form* 23 *February* 1994)

Abstract—A solution of a quasi-steady thermoelastic problem for the elastic convective half-space when a heat source with constant power is moving on its surface is obtained. The method of Fourier transformation is used for obtaining an analytical expansions for the temperature, the thermal stressses and the displacements. An asymptotic formula for finding the normal dispalcements of the half-space boundary is constructed. Numerical results are presented in the form of figures for the temperature and the normal displacements.

1. INTRODUCTION

THERE are a number of technological processes in which heat exchange is realized simultaneously all over the surface with heating; for example, induction heating with high frequency, electrospark alloying, etc. To analyse temperature fields and thermal distortion of the surface in such processes it is necessary to solve a boundary problem of thermoelasticity under the condition of heat exchange on all of the surface, including below the source [1]. Of the various methods for solving boundary problems of thermoelasticity, the most direct is to write down the governing integral equations in terms of appropriate Green's functions. Several approaches to define these functions exist.

The concept of a sinusoidal temperature wave moving uniformly on the surface of an elastic half-space together with the use of the thermoelastic potential function of displacements are presented in the papers of scientific group directed by R. Burton (USA) [2, 3]. The considerable difficulties in the calculation of the summation of slowly convergent Fourier's series should be taken as a defect of the method.

In ref. [4], on the basis of the solution of a thermal conduction problem for a momentary heat source acting on the surface of elastic half-space [5], analytical expressions of quasi-steady-space displacements and tangential stresses suitable for the arbitrary values of Peclet's parameter were obtained. Corresponding values inside the half-space were found in ref. [6].

An asymptotic solution for large (>10) Peclet numbers determining the distribution of heat flux in each contacting body has been constructed in ref. [7]. The application of the finite-element method to this aim has been realized in ref. [8].

In all of the aforementioned articles it is assumed that the surface of the half-space outside the heat area is heat insulated. The solution of the quasi-steady problem for a heat source moving uniformly on the boundary of the half-space, taking into account heat exchange with the outward environment by Newton's law, has been obtained in ref. [9]. Corresponding thermal stresses and displacements in the case of large Peclet numbers were determined in ref. [10].

The approximate significances of temperature and displacements in the elastic half-space heated by a heat source moving uniformly on its boundary with the heat flux distributing on the final segment were found in ref. [11] on the basis of the integral characters method [12].

The aim of the present article is the development of the methods offered in refs. [6, 9] to find fundamental and thermoelastic solutions for a line of heat sources with constant power (plane strain) moving with a constant velocity on the elastic half-space surface. The problem is formulated in the limits of classic linear thermoelastic theory. It is assumed that the material of the half-space is homogeneous and isotropic, and its physical properties do not depend on temperature. We neglect inertial effects and the influence of thermoelastic coupling.

2. THERMAL-CONDUCTION PROBLEM

With the given assumptions, the thermal-conduction equation in the coordinate axes $x0y$ rigidly

NOMENCLATURE

h	coefficient of heat exchange
k	thermal diffusivity
K	thermal conductivity
Q	intensity of heat flux generation
r	$(x^2+y^2)^{1/2}$, equation (3)
$\tilde{R}(x,y,s)$	function defined in equation (15)
s, s_0	values defined in equations (12) and (13)
$T(x,y)$	temperature
$T^*(x,y)$	dimensionless temperature, equation (19)
$T_{,x}$	$= \partial T/\partial x$
$T_{,y}$	$= \partial T/\partial y$
u, v	displacements
V	velocity of the moving heat source
x, y	Cartesian coordinate pair
X	stress function.

Greek symbols

α_T	coefficient of linear temperature expansion
α	$= \gamma/\beta$
β	$= V/k$
γ	$= h/K$
$\delta(x)$	Dirac function
Δ	$\partial^2/\partial x^2+\partial^2/\partial y^2$, Laplace operator
$\eta(\xi)$	function defined in equation (5)
μ	shear modulus
ν	Poisson ratio
ξ	parameter of Fourier integral transform
$\sigma_x, \sigma_y, \sigma_{xy}$	components of temperature stresses
Φ	Airy function
Ψ	potential of displacements.

connected with a source moving with constant velocity V on the boundary $y = 0$ of the half-space has the form

$$\Delta T+\beta T_{,x} = 0 \quad |x| < \infty \quad y \geqslant 0. \tag{1}$$

We construct the solution of differential equation (1), satisfying the boundary conditions

$$KT_{,y}-hT = -Q\delta(x) \quad |x| < \infty \quad y = 0, \tag{2}$$

$$T, T_{,x}, T_{,y} \to 0 \quad \text{when } r \equiv (x^2+y^2)^{1/2} \to \infty. \tag{3}$$

Applying Fourier integral transforms with respect to x to equation (1) and boundary conditions (2), (3) gives

$$\bar{T}(\xi,y) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} T(x,y) \times \exp(-\mathrm{i}\xi x)\,\mathrm{d}x \quad |x| < \infty \quad y \geqslant 0, \tag{4}$$

and we obtain

$$\bar{T}_{,yy}-\eta^2\bar{T} = 0 \quad \eta(\xi) \equiv \sqrt{(\xi^2-\mathrm{i}\beta\xi)} \quad y \geqslant 0, \tag{5}$$

$$K\bar{T}_{,y}-h\bar{T} = (2\pi)^{1/2}Q \quad y = 0. \tag{6}$$

The solution of the differential equation (5) with condition (6) has the form

$$\bar{T}(\xi,y) = \frac{Q}{\sqrt{(2\pi)}K}\frac{\exp[-y\eta(\xi)]}{\gamma+\eta(\xi)}. \tag{7}$$

By means of Fourier integral transformation

$$T(x,y) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} \bar{T}(\xi,y)\exp(\mathrm{i}\xi x)\,\mathrm{d}\xi \tag{8}$$

and from equation (7) we see that

$$T(x,y) = \frac{Q}{2\pi K}\int_{-\infty}^{\infty}\frac{\exp[-y\eta(\xi)+\mathrm{i}\xi x]}{\gamma+\eta(\xi)}\,\mathrm{d}\xi \quad |x| < \infty \quad y \geqslant 0. \tag{9}$$

In relationship (9) for the satisfaction of conditions (3) a positive branch of the function of many figures $\eta(\xi)$ was chosen.

We present solution (9) in the form

$$T(x,y) \equiv \frac{Q}{\pi K}\operatorname{Re} T^* = \frac{Q}{\pi K}\operatorname{Re}\left(\int_0^{\infty}\frac{\exp[-y\eta(\xi)+\mathrm{i}\xi x]}{\gamma+\eta(\xi)}\,\mathrm{d}\xi\right). \tag{10}$$

In expression (10) the integral may be simplified if we use integration along the closed loops [6]:

$$\Gamma^{\pm} = L^{\pm}\cup C_R^{\pm}\cup C^{\pm}\cup I\cup C_\varepsilon \tag{11}$$

as is plotted in Fig. 1. Here indices '$\pm$' mean integration along the curve Γ^+ at $x > 0$ and Γ^- when $x < 0$. The integrand in relation (10) is analytical inside $\Gamma^\pm$. The branch points are $\xi = 0$ and $\xi = \mathrm{i}\beta$; corresponding sections have been conducted along negative real and postive imaginary axes in such a way that a branch of the function $\eta(\xi)$ receives a positive significance on the positive part of the real axis.

Integrals along $C_R^{\pm}$ and C_ε at $R \to \infty$ and $\varepsilon \to 0$ are equal to zero. On the curves $C^\pm$ an argument of the exponential function in the integrand (10) receives a real negative value $-s$, where

$$s = y\eta(\xi)-\mathrm{i}\xi x \quad s_0 \leqslant s \leqslant \infty, \tag{12}$$

$$s_0 = 0.5\beta(x+\sqrt{(x^2+y^2)}). \tag{13}$$

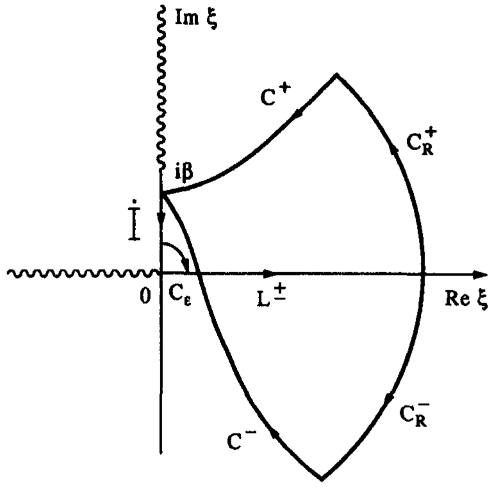

FIG. 1. Contour of integration.

From (12) we see that along the curves $C^{\pm}$ the variable ξ takes the meaning

$$\xi(s) = \frac{\mathrm{i}(\beta y^2 + 2xs) + y\sqrt{(4s^2 - 4\beta xs - \beta^2 y^2)}}{2(x^2 + y^2)}$$

while on the axis Im ξ we have

$$\xi(s_0) = 0.5\mathrm{i}\beta[1 + x(x^2 + y^2)^{-1/2}] < \mathrm{i}\beta \text{ at } y > 0. \tag{14}$$

On the segment I, where $\xi = \mathrm{i}\tau$, $0 < \varepsilon \leqslant \tau \leqslant \xi(s_0)/\mathrm{i}$, the integral (10) takes a wholly imaginary value and, therefore, cannot give a contribution to $T(x, y)$.

Thus on the basis of the Cauchy theorem, as a result of integration along the contours $\Gamma^{\pm}$ (11) from (10) we find the temperature in an arbitrary point of half-space

$$T(x,y) = \frac{2Q}{\pi K}\int_{s_0}^{\infty} \frac{\tilde{R}(x,y,s)\exp(-s)}{\sqrt{(4s^2 - 4\beta xs - \beta^2 y^2)}}\,\mathrm{d}s$$

$$|x| < \infty \quad y \geqslant 0$$

$$\tilde{R}(x,y,s) = \frac{2(s^2 - \beta xs) + \gamma y(2s - \beta x)}{2[s^2 - \beta xs + \gamma y(2s - \beta x) + \gamma^2(x^2 + y^2)]}. \tag{15}$$

Supposing in (15) $s^* = s + s_0$, we obtain definitively (we omit asterisks)

$$T(x,y) = \frac{Q}{\pi K}\exp(-s_0)\int_0^{\infty} \frac{R(x,y,s)\exp(-s)}{\sqrt{[s^2 + (2s_0 - \beta x)s]}}\,\mathrm{d}s$$

$$R(x,y,s) \equiv \tilde{R}(x,y,s+s_0) = \frac{s^2 + a_1 s + b_1}{s^2 + a_2 s + b_2}$$

$$a_1 = \beta r + \gamma y, \quad a_2 = \beta r + 2\gamma y,$$

$$b_1 = (\beta y/2)^2 + \gamma\beta yr/2, \quad b_2 = (\beta y/2)^2 + \gamma\beta yr + \gamma^2 r^2. \tag{16}$$

We consider two known property cases of solution (16). Let $h = 0$, $y \geqslant 0$. Then $\gamma = 0$, $R(x,y,s) = 1$ and

$$T(x,y) = \frac{Q}{\pi K}\exp(-s_0)\int_0^{\infty} \frac{\exp(-s)\,\mathrm{d}s}{\sqrt{[s^2 + (2s_0 - \beta x)s]}}. \tag{17}$$

By means of formula 3.364.3 in ref. [13] we can calculate the integral in relation (17). We have

$$\int_0^{\infty} \frac{\exp(-s)\,\mathrm{d}s}{\sqrt{[s^2 + (2s_0 - \beta x)s]}} = \exp(\beta r/2)K_0(\beta r/2)$$

and the temperature of half-space, when the heat source moves uniformly on its insulated surface, is

$$T(x,y) = \frac{Q}{\pi K}\exp(-\beta x/2)K_0(\beta r/2). \tag{18}$$

Relation (18) was first obtained in article [5]. $K_0(*)$ is a modified Bessel function of the second kind.

Let $h \neq 0$, $y = 0$. Then, from (13) we have that $s_0 = \beta x$ at $x > 0$ and $s_0 = 0$ at $x < 0$. The function $R(x,y,s)$ has the form

$$R(x,y,s) = \frac{s^2 + \beta|x|s}{s^2 + \beta|x|s + \gamma^2 x^2}$$

and the temperature of the boundary points of the convective cooling half-space is

$$T(x,0) = \frac{Q}{\pi K}\exp(-s_0)\int_0^{\infty} \frac{\sqrt{(s^2 + \beta|x|s)}\exp(-s)\,\mathrm{d}s}{s^2 + \beta|x|s + \gamma^2 x^2}$$

$$= \frac{Q}{\pi K}T^*(x,0)$$

$$= \frac{Q}{\pi K}\exp(-s_0)\int_0^{\infty} \frac{\sqrt{(s^2 + s)}\exp[-\beta|x|s]\,\mathrm{d}s}{s^2 + s + \alpha^2}$$

$$\alpha = \gamma/\beta. \tag{19}$$

Relation (19) coincides completely with the results of article [9].

3. THERMOELASTIC PROBLEM

The components of temperature stresses are presented through the Airy function Φ and the thermoelastic potential of displacements Ψ in the form [14]

$$\sigma_x = X_{,yy} \quad \sigma_y = X_{,xx} \quad \sigma_{xy} = -X_{,xy} \tag{20}$$

$$X \equiv \Phi - 2\mu\Psi. \tag{21}$$

We obtain functions Φ and Ψ from the solution of the boundary thermoelastic problem:

$$\Delta\Delta\Phi = 0 \quad \Delta\Psi = NT \quad |x| < \infty \quad y \geqslant 0 \tag{22}$$

$$\sigma_y(x,0) = \sigma_{xy}(x,0) = 0 \quad |x| < \infty \tag{23}$$

$$\sigma_x, \sigma_y, \sigma_{xy} \to 0 \quad \text{at} \quad r \to \infty. \tag{24}$$

Here $N = (1+\nu)\alpha_T/(1-\nu)$.

Equations (22)–(24) are transformed in the transform space (4) of the Fourier integral transformation with respect to variable x:

$$\left(\frac{d^2}{dy^2} - \xi^2\right)^2 \bar{\Phi} = 0 \quad \left(\frac{d^2}{dy^2} - \xi^2\right)\bar{\Psi} = N\bar{T}, \tag{25}$$

$$\bar{\sigma}_y(\xi, 0) = \bar{\sigma}_{xy}(\xi, 0) = 0. \tag{26}$$

Usually the infinite solutions of the differential equations (25) and (26) have the form

$$\bar{\Phi}(\xi, y) = \frac{2\mu N}{i\beta\xi}[-1 + y(-|\xi| + \eta(\xi))] \times \exp(-y|\xi|)\bar{T}(\xi, 0),$$

$$\bar{\Psi}(\xi, y) = -\frac{N}{i\beta\xi}\exp(-y\eta(\xi))\bar{T}(\xi, 0) \tag{27}$$

where $\bar{T}$ is the Fourier transform of the temperature T (12). On the basis of (27) we find the Fourier transformation of the Airy function (21)

$$\bar{X}(\xi, y) = -\frac{2\mu N}{i\beta\xi}\{[1 - y(-|\xi| + \eta(\xi))]\exp(-y|\xi|) - \exp[-y\eta(\xi)]\}\bar{T}(\xi, 0). \tag{28}$$

We introduce the function

$$L = L_1 + y(L_2 + L_3), \tag{29}$$

where

$$L_1(x, y) \equiv \frac{Q}{\pi K}\operatorname{Re} L_1^*(x, y) = \frac{Q}{\pi K}\operatorname{Re}\left\{i\int_0^\infty \frac{\exp[-\xi(y - ix)]\,d\xi}{\xi[y + \eta(\xi)]}\right\} \tag{30}$$

$$L_2(x, y) \equiv \frac{Q}{\pi K}\operatorname{Re} L_2^*(x, y) = \frac{Q}{\pi K}\operatorname{Re}\left\{-i\int_0^\infty \frac{\eta(\xi)\exp[-\xi(y - ix)]\,d\xi}{\xi[y + \eta(\xi)]}\right\} \tag{31}$$

$$L_3(x, y) \equiv \frac{Q}{\pi K}\operatorname{Re} L_3^*(x, y) = \frac{Q}{\pi K}\operatorname{Re}\left\{i\int_0^\infty \frac{\exp[-\xi(y - ix)]\,d\xi}{y + \eta(\xi)}\right\} \tag{32}$$

$$M(x, y) \equiv \frac{Q}{\pi K}\operatorname{Re} M^*(x, y) = \frac{Q}{\pi K}\operatorname{Re}\left\{-i\int_0^\infty \frac{\exp[-y\eta(\xi) + i\xi x]\,d\xi}{\xi[y + \eta(\xi)]}\right\}. \tag{33}$$

Applying the inversion formula of the Fourier integral transform (8) to relationship (28), and taking into consideration equations (28)–(33), we obtain

$$X(x, y) = \frac{2\mu N}{\beta}[L(x, y) + M(x, y)]. \tag{34}$$

From formula (34) it follows that the stress function X is a sum of two integrals. The first of them, L, corresponds to a biharmonic function Φ and has the form of transformation of the Laplace integral transform with respect to variable ξ of some function at $y \geqslant 0$. The second integral, M, is connected with the thermoelastic displacements potential Ψ:

$$\Psi(x, y) = \frac{2\mu N}{\beta}M(x, y) \tag{35}$$

where

$$M_{,x} = T \quad M_{,yy} = -T_{,x} - \beta T. \tag{36}$$

4. STRESSES

We denote

$$y(L_{2,xx}^* + L_{3,xx}^*) \equiv S_1,$$
$$iL_{3,x}^* = L_{3,y}^* = L_{1,xx}^* = -L_{1,xy}^* = -iL_{1,yy}^* \equiv S_2,$$
$$L_{2,x}^* = -iL_{2,y}^* \equiv S_3. \tag{37}$$

Differentiating the stress function X (34) according to equation (20), and taking into account relations (35)–(37), we find

$$\sigma_x = N_1 \operatorname{Re}(-S_1 + S_2 + 2iS_3 - \beta T^* - T_{,x}^*)$$
$$\sigma_y = N_1 \operatorname{Re}(S_1 + S_2 + T_{,x}^*)$$
$$\sigma_{xy} = -N_1 \operatorname{Re}(iS_1 + S_3 + T_{,y}^*) \quad N_1 = 2\mu N Q(\pi K\beta)^{-1}. \tag{38}$$

We note that at $y = 0$ from (38) it follows that $\sigma_y = \sigma_{xy} = 0$.

5. DISPLACEMENTS

Elastic displacements are connected with temperature stresses by the formulae of Duamel–Nayman [15]

$$2\mu u_{,x} = (1-\nu)\sigma_x - \nu\sigma_y + 2\mu(1+\nu)\alpha_T T$$
$$2\mu v_{,y} = (1-\nu)\sigma_y - \nu\sigma_x + 2\mu(1+\nu)\alpha_T T. \tag{39}$$

Substituting the value of stresses (38) into the right-hand sides of (39), and integrating with respect to x and y accordingly, we obtain

$$u = N_2 \operatorname{Re}\{(1-\nu)i[2L_2^* + L_3^* + yL_{23}^*] - \nu L_{,x}^* - T^*\}$$
$$v = N_2 \operatorname{Re}\{(1-\nu)[L_2^* + 2L_3^* - yL_{23}^*] - \nu L_{,y}^* - M_{,y}^*\}. \tag{40}$$

Here

$$N_2 = N_1/(2\mu) \quad L_{23}^* = S_2 + iS_3$$
$$L_{,x}^* = iS_3 - iyL_{23}^* \quad L_{,y}^* = L_2^* + yL_{23}^*.$$

6. FINDING OF THE FUNCTIONS L_2^*, L_3^* AND $M_{,y}^*$

It follows from (38) and (40) that the components of the tensor of temperature stresses and the vector of displacements may be expressed through the functions L_2^*, L_3^*, T^*, their derivatives, and also the integral $M_{,y}^*$. The temperature T^* in any point of the half-space is given by formula (16). Now we consider integrals $L_j^*, j = 2, 3$ ((31) and (32)). We denote

$$s \equiv \xi(y-\mathrm{i}x),$$
$$2s_{\pm}^2 \equiv \pm(s^2-\beta xs)+\sqrt{[(s^2-\beta xs)^2+\beta^2y^2]}.$$

Then

$$\sqrt{(\xi^2-\mathrm{i}\beta\xi)} = \frac{s_+-\mathrm{i}s_-}{(y-\mathrm{i}x)}$$

and functions L_2^*, L_3^* have the form of transformation of the integral Laplace transform with transform parameter $p = 1$:

$$L_j^*(x,y) = \int_0^\infty \bar{L}_j(x,y,s)\,\mathrm{e}^{-s}\,\mathrm{d}s \quad \bar{L}_j = A_j/D \quad j = 2,3$$

$$sA_2 = \gamma(xs_+-ys_-)-\mathrm{i}[s_+(s_++\gamma y)+s_-(s_-+\gamma x)]$$
$$A_3 = -(s_-+\gamma x)+\mathrm{i}(s_++\gamma y)$$
$$D = (s_++\gamma y)^2+(s_-+\gamma x)^2. \qquad (41)$$

According to (33) the integral $M_{,y}^*$ is

$$M_{,y}^* = \mathrm{i}\int_0^\infty \frac{\eta(\xi)\exp[-y\eta(\xi)+\mathrm{i}\xi x)]\,\mathrm{d}\xi}{\xi[\gamma+\eta(\xi)]}$$

which we shall reduce to a form comfortable for calculations with the help of integration along the closed curves (11). We obtain

$$M_{,y}^* = \int_{s_0}^\infty \bar{M}_{,y}(x,y,s)\exp(-s)\,\mathrm{d}s$$

$$\bar{M}_{,y}(x,y,s) = \frac{(-\gamma\xi_-+\mathrm{i}(\xi_+^2+\xi_-^2+\gamma\xi_+))(\sqrt{(4s^2-4\beta xs-\beta^2y^2)}-\mathrm{i}\beta y)}{s((\gamma+\xi_+^2)+\xi_-^2)\sqrt{(4s^2-4\beta xs-\beta^2y^2)}}$$

$$\xi_+ = \frac{y(2s-\beta x)}{2(x^2+y^2)} \quad \xi_- = \frac{x\sqrt{(4s^2-4\beta xs-\beta^2y^2)}}{2(x^2+y^2)}$$

$$\sqrt{(\xi^2-\mathrm{i}\beta\xi)} = \xi_++\mathrm{i}\xi_-. \qquad (42)$$

If the surface of half-space is insulated ($h = 0, \gamma = 0$), then from (41) it follows that

$$\bar{L}_2 = -\mathrm{i}s^{-1} \quad \bar{L}_3 = \mathrm{i}[s^2-\mathrm{i}\beta(y-\mathrm{i}x)s]^{-1/2}.$$

Hence

$$L_2^* = -\mathrm{i}\int_0^\infty \frac{\exp(-s)\,\mathrm{d}s}{s} = -\mathrm{i}\int_0^\infty \frac{\exp[-\xi(y-\mathrm{i}x)]\,\mathrm{d}\xi}{\xi}$$

$$= -\mathrm{i}\int_{y-\mathrm{i}x}^\infty \frac{\mathrm{d}s}{s} = -\mathrm{i}\ln(s)|_{y-\mathrm{i}x}^\infty \qquad (43)$$

$$L_3^* = \mathrm{i}\int_0^\infty \frac{\exp(-s)\,\mathrm{d}s}{\sqrt{(s^2-\mathrm{i}\beta(y-\mathrm{i}x)s)}}. \qquad (44)$$

We note that function L_2^*, although non-limited, is differentiated. By means of formula (13) from p. 128 of ref. [16] we calculate the integral in relation (44):

$$L_3^* = \mathrm{i}\exp[-\mathrm{i}\beta(y-\mathrm{i}x)/2]K_0[-\mathrm{i}\beta(y-\mathrm{i}x)/2].$$

Taking into account [17]

$$K_0(z) = 0.5\pi\mathrm{i}H_0^{(1)}(\mathrm{i}z) \quad -\pi < \arg(z) < 0$$

and from (44) we find

$$L_3^* = -0.5\pi\exp[-\mathrm{i}\beta(y-\mathrm{i}x)/2]H_0^{(1)}[\beta(y-\mathrm{i}x)/2]. \qquad (45)$$

$H_0^{(1)}$ is a Hankel function.

The real part of the function $M_{,y}^*$ at $\gamma = 0$, $y \neq 0$ from (41) has the form

$$\operatorname{Re}M_{,y}^* = -\frac{\pi}{2}+\frac{\beta y}{2}\exp(-s_0)$$
$$\times\int_0^\infty \frac{\exp(-s)\,\mathrm{d}s}{(s+s_0)\sqrt{[s^2+(2s_0-\beta x)s]}}$$
$$= -\frac{\pi}{2}+\cos^{-1}(x/r)-\frac{\beta y}{2}$$
$$\times\int_0^1 \exp(-\beta xs/2)K_0(\beta rs/2)\,\mathrm{d}s \qquad (46)$$

Formulae (45), (46) were obtained for the first time in ref. [6].

If $\gamma \neq 0$ and $y = 0$ then from (41) we obtain

$$L_2^* = -\mathrm{i}\left[\alpha\int_0^1 \frac{\sqrt{(s-s^2)}\exp(-\beta xs)\,\mathrm{d}s}{s(s^2-s+\alpha^2)}\right.$$
$$\left.+\int_0^\infty \frac{(s-1)\exp(-\beta xs)\,\mathrm{d}s}{s^2-s+\alpha^2}\right]$$
$$+\alpha\exp(-\beta x)\int_0^\infty \frac{\exp(-\beta xs)\,\mathrm{d}s}{\sqrt{(s^2+s)(s^2+s+\alpha^2)}}$$

if $x > 0$

$$L_2^* = -\mathrm{i}\int_0^\infty \frac{(s-1)\exp(-\beta|x|s)\,\mathrm{d}s}{s^2-s+\alpha^2}$$
$$-\alpha\int_0^\infty \frac{(s+1)\exp(-\beta|x|s)\,\mathrm{d}s}{\sqrt{(s^2+s)(s^2+s+\alpha^2)}}$$

if $x < 0$. (47)

For the function L_3^* the corresponding functions will take the form

$$L_3^* = -\int_0^1 \frac{\exp(-\beta x s)\,ds}{\sqrt{(s^2+s)}+\alpha} - \alpha\exp(-\beta x)\int_0^\infty \frac{\exp(-\beta x s)\,ds}{s^2+s+\alpha^2} + i\exp(-\beta x)\int_0^\infty \frac{\sqrt{(s^2+s)}\exp(-\beta x s)\,ds}{s^2+s+\alpha^2} \quad \text{if } x>0 \tag{48}$$

and

$$L_3^* = i\int_0^\infty \frac{\sqrt{(s^2+s)}\exp(-\beta|x|s)\,ds}{s^2+s+\alpha^2} + \alpha\int_0^\infty \frac{\exp(-\beta|x|s)\,ds}{s^2+s+\alpha^2} \quad \text{if } x<0.$$

The real part of the function M_y^* in the present case is

$$\operatorname{Re} M_y^* = -L_2^*. \tag{49}$$

7. ASYMPTOTIC BEHAVIOUR

We shall find a normal thermal displacement of the surface of a half-space. From (40) we have

$$v(x,0) = 2(1-\nu)N_2 \operatorname{Re}[L_2^* + L_3^*]. \tag{50}$$

Hence, in view of formulae (47)–(49), we obtain

$$v(x,0) = 2(1-\nu)N_2 V(x) \tag{51}$$

$$V(x) = H(x)\int_0^1 \frac{\exp(-\beta x s)\,ds}{\sqrt{(s-s^2)}+\alpha} - \alpha\begin{cases}\exp(-\beta x)\; V_+(x), & x>0\\ V_-(x), & x<0\end{cases} \tag{52}$$

$$V_+(x) = \int_0^\infty \frac{(s-\sqrt{(s^2+s)})\exp(-\beta x s)\,ds}{\sqrt{(s^2+s)}(s^2+s+\alpha^2)} \tag{53}$$

$$V_-(x) = \int_0^\infty \frac{(\sqrt{(s^2+s)}-s-1)\exp(\beta x s)\,ds}{\sqrt{(s^2+s)}(s^2+s+\alpha^2)}. \tag{54}$$

Integrands in expressions (53) and (54) are rapidly convergent at $s\to\infty$, because the main contribution to $V_\pm(x)$ will be determined by the conduct of this function in the vicinity of zero. Considering that $0\leqslant s\leqslant\delta$, $\delta\ll 1$, and $\alpha<\sqrt{\delta}$, we find

$$V_+(x) \simeq J_2(x,\delta)-J_1(x,\delta),$$
$$V_-(x) \simeq J_1(x,\delta)-J_2(x,\delta)-J_3(x,\delta). \tag{55}$$

Here

$$J_1(x,\delta) = \ln|\alpha^2+\delta| - \ln|\alpha^2| + \beta|x| \times [\delta-\alpha^2\ln|\alpha^2+\delta| + \alpha^2\ln|\alpha^2|]$$

$$J_2(x,\delta) = 2[\delta-\alpha\tan^{-1}(\delta^{1/2}/\alpha)] + 2\beta|x|[\delta^{3/2}/3-\alpha^2\delta^{1/2}+\alpha^3\tan^{-1}(\delta^{1/2}/\alpha)]$$

$$J_3(x,\delta) = 2\tan^{-1}(\delta^{1/2}/\alpha)/\alpha + 2\beta|x|[\delta-\alpha\tan^{-1}(\delta^{1/2}/\alpha)]. \tag{56}$$

At $\alpha\to 0$ from relations (55) and (56) it follows that $V_+(x)=0$, $V_-(x)=-\pi/\alpha$. Since [13]

$$\int_0^1 \frac{\exp(-\beta|x|s)\,ds}{\sqrt{(s-s^2)}} = \pi\exp(-\beta x/2)I_0(\beta|x|/2)$$

then

$$V(x) = \begin{cases}\pi\exp(-\beta x/2)I_0(\beta|x|/2) & x>0\\ \pi & x<0.\end{cases} \tag{57}$$

$I_0(*)$ is a modified Bessel function of the first kind. At $V(x)$ relation (51) determined by formula (57) coincides completely with the result given in ref. [4].

8. NUMERICAL ANALYSIS

The dependence of the distribution of non-dimensionalized temperature $T^*(x,0)$ of the elastic half-space surface calculated by formula (19) on the dimensionless parameter βx at $\alpha = 0.01, 0.1, 1, 10$ is shown in Fig. 2. We see that with the growth of heating from the surface of a half-limited body its temperature falls. We note also that presented by the curve $\alpha = 0.01$ the results coincide with the facts, obtained by formula (18), to within 10^{-4} in the case where the surface $y=0$ of the half-space is insulated.

The dependence of the change of the dimensionless normal displacement $V(x)$ of the boundary of the half-space, found by the formulae (51)–(54), on βx is presented in Fig. 3. The results, presented by dotted segments, have been obtained by calculation of formula (57). It is seen that the value α of order 0.01 leads to results little distinguished from the case of the insulated surface of the half-space. It should be noted that, at small (of order 0.1 and less) values of the parameter α, in order to calculate normal dis-

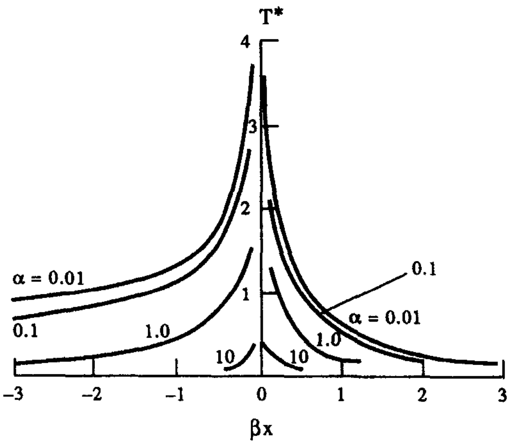

FIG. 2. Variations of dimensionless temperature T^* with βx for four values of parameter $\alpha = 0.01, 0.1, 1, 10$.

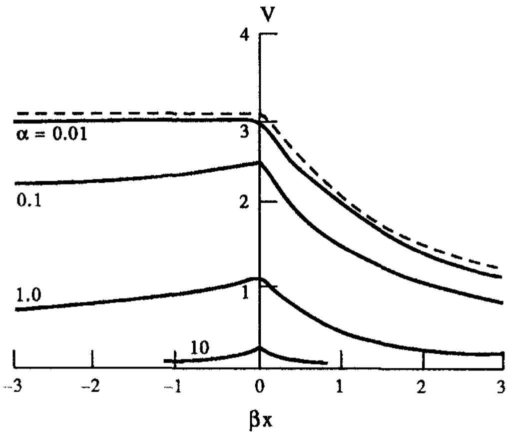

FIG. 3. The change of the normal displacements $V(x)$ of the boundary of the half-space with βx for $\alpha = 0.01, 0.1, 1, 10$.

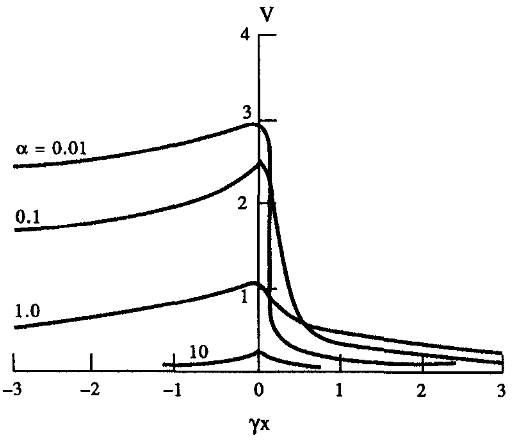

FIG. 4. The change of $V(x)$ with γx for $\alpha = 0.01, 0.1, 1, 10$.

placements we can use the asymptotic relations (52), (55) and (56).

The dependence of the change of $V(x)$ on the dimensionless parameter γx is plotted in Fig. 4. In the presence of convective heating at the entrance of the input zone ($x < 0$), the behaviour of temperature displacements of the surface of the half-plane differs from that of displacements at $\gamma = 0$. In ref. [1] it was shown that with the increase of velocity for $h = 0$ the thermal conductivity before the source becomes less effective and the displacements fall in the input zone. It is ascertained that the thermal distortions in the input zone increase with the growth of β. At $x \geqslant 0$ the displacements increase with decrease of the moving source velocity. At $\alpha < 0.1$ it is possible to make the calculation using formula (57). Here with the growth of parameter β ($\geqslant 10$) the conduct of displacements $V(x)$ may be approximated by the Heavyside function $H(-x)$ with greater accuracy.

Acknowledgement—The authors are pleased to acknowledge the support of the National Committee of Science and Technology of Ukraine through grant number 1/904.

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